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Antiferromagnetic spin ladders—the case of odd numbers of spin-$s = \frac{1}{2}$ chains

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Abstract. The problem of three coupled spin-$s = \frac{1}{2}$ antiferromagnetic chains has been investigated. Using the bosonization method we find that, regardless of the sign of the interchain coupling, the low-energy excitations are gapless. We show that the low-energy excitations are equivalent to those of the single antiferromagnetic $s = \frac{1}{2}$ Heisenberg chain.

1. Introduction

It already known [1] that one-dimensional Heisenberg chains with half-integer spin are gapless and those with integer spin are gapped. The discovery of high-T_c superconductivity and, in particular, the spin-$s = \frac{1}{2}$ spin ladders [2, 3] have renewed interest in low-dimensional antiferromagnetism. The behaviour of the $s = \frac{1}{2}$ ladders is fundamentally different for even and odd numbers of coupled chains [4]. Numerically, the difference between odd and even chains is well understood [4–6]. Theoretically, it has been shown [7, 8] within the coherent-state representation for spins that the difference between ladders composed of even and odd numbers of spin chains is due to the difference in the topological Pointryagin index θ ($\theta = 2\pi s n_e$; s is the spin $s = \frac{1}{2}$, and n_e is the number of chains). This identification [8] is obtained within the semiclassical limit $s \gg 1$. An alternative method which has been used for the spin-$\frac{1}{2}$ antiferromagnets in one dimension is the bosonization method [9–11]. Recently the bosonization method has been used to study the case of two antiferromagnetic spin ladders [12]. Here we consider the natural extension to the case of ladders composed of odd numbers of spin chains.

The purpose of this paper is to show that for an odd number of weakly coupled antiferromagnetic (AF) chains the low-energy spin excitations are dominated by gapless spin excitations. Explicitly, we consider three AF spin chains with an intrachain exchange coupling $J_\parallel$ and an interchain coupling $\pm J_\perp$, $J_\parallel > |J_\perp|$. The fact that the two-chain spin ladders have massive spin excitations enables us to integrate them out and to obtain an effective action for the lowest spin mode. The lowest spin mode is gapless and is similar to the spin-$\frac{1}{2}$ spinon with a spectrum like that of the $s = \frac{1}{2}$ Heisenberg chain.

The importance of the results and the method consists in the fact that one finds massless excitations, contrary to the existing results found for the case of three Hubbard chains [13, 14]. The crucial point in our method is that we have been able to separate the massive modes from a massless one. We do not perform an orthogonal transformation which mixes the modes as in reference [3]. We use the fact that a pair of chains forms an $s = 1$ massive system. The massive $s = 1$ system couples to the third chain leaving the excitations massless. Unlike the spin-coherent-state formalism method [7, 8], our method is suitable for spin and charge excitations. This fact is relevant for related problems like that of the spin-striped cuprates.

The methodology of this paper is as follows. We introduce the Jordan–Wigner (JW) transformation which maps the spins in each chain into spinless fermions [9]. As a result one obtains three coupled sine–Gordon models described in terms of the bosonic field $\theta_r(x)$ and the dual field $\phi_r(x)$, $r = -1, 0, 1$. We form a spin-1 pair by combining the chain $r = 1$ with the chain $r = -1$. One obtains massive excitations for the pairs

$$\theta_\pm(x) = \frac{1}{\sqrt{2}}(\theta_{r=1}(x) \pm \theta_{r=-1}(x)).$$

We project the massive modes $\theta_\pm(x)$ and their dual $\phi_\pm(x)$ and obtain an effective Hamiltonian for the chain $r = 0$. The projection of the $s = 1$ modes is done using the mapping of each sine–Gordon mode to two coupled Ising models [15–17]. The fact that the correlation functions and susceptibility are known for the classical two-dimensional Ising model [15] allows us to perform an exact integration for the massive modes.

The plan of this paper is as follows. Section 2 is devoted to the presentation of the model, the case of three $s = \frac{1}{2}$ chains. Section 3 discusses the gaps for the two-chain case. Section 4 is devoted to the projection of the massive modes. As a result we obtain the effective Hamiltonian for the low-energy excitations for the three-chain case. Section 5 is devoted to explicit calculations made with the help of the Ising correlation functions.

2. The Jordan–Wigner bosonization for the three-spin-chain problem

The model for the three-spin-chain AF ladder is

$$H = J_\parallel \sum_{r=-1,0,1} \sum_x \boldsymbol{S}_r(x) \cdot \boldsymbol{S}_r(x+a) + J_\perp \sum_{r=-1,0,1} \sum_x \boldsymbol{S}_r(x) \cdot \boldsymbol{S}_{r+1}(x). \qquad (1)$$

$J_\parallel > 0$ is the AF intrachain coupling and $\pm J_\perp$ is the interchain coupling which obeys $J_\parallel > |J_\perp|$. r is the index labelling the chains; $r = -1, 0, 1$. We consider first the pair of chains $r = 1$ and $r = -1$ coupled by the interchain coupling $J_\perp$. In agreement with references [12] and [22] we find that the pair $r = 1$ and $r = -1$ forms an $s = 1$ massive spin system. When the third chain $r = 0$ is added to the system it couples to the $s = 1$ massive system (the pair $r = 1$ and $r = -1$) via the interchain coupling $J_\perp$. We construct a low-energy Hamiltonian for the three-chain problem by integrating out the massive modes with $s = 1$ obtained from the pair $r = 1$ and $r = -1$. As a result, the Heisenberg chain $r = 0$ becomes an effective Heisenberg Hamiltonian with gapless excitations.

In the second part of this section we bosonize the model given in equation (1). In order to be explicit we will separate the interchain Hamiltonian such that the massive spin pair is formed between the chains $r = 1$ and $r = -1$. Using the JW transformation we map the spin-half operators into hard-core bosons and next into spinless fermions [7]. Since the hard-core bosons in different chains commute we do not need to introduce a Klein factor in order to have anticommutation between chains [18]. The fermions in each chain are expressed in terms of the boson fields $\theta_r(x)$ and their duals $\phi_r(x)$. These fields obey the commutation relations

$$[\theta_r(x), \phi_{r'}(x')] = -\frac{\mathrm{i}}{2}\delta_{r,r'}\,\mathrm{sgn}(x - x').$$

We will use the fields $\theta_r(x)$ and $\phi_r(x)$ or the chiral representation

$$\theta_{R,r}(x) = \frac{1}{\sqrt{2}}(\theta_r(x) - \phi_r(x))$$

and

$$\theta_{L,r}(x) = \frac{1}{\sqrt{2}}(\theta_r(x) + \phi_r(x)).$$

The anticommutation of the left and right fermions is achieved by using a set of real Majorana fermions [19, 20], $\alpha_R = \tau_x$ and $\alpha_L = -\mathrm{i}\tau_y$ (τ_x and τ_y are Pauli matrices, $\alpha_R\alpha_L + \alpha_L\alpha_R = 0$, $\alpha_R^2 = \alpha_L^2 = 1$, $-\alpha_R\alpha_L = \tau_z$ with τ_z being a conserved quantity). We obtain the following representation for the spin-$\frac{1}{2}$ AF chains:

$$S_r^z(x) = \frac{1}{\sqrt{\pi}}\,\partial_x\theta_r(x) + \tau_z\frac{(-1)^x}{2\pi a}[\mathrm{e}^{-\mathrm{i}\sqrt{4\pi}\theta_{R,r}(x)}\mathrm{e}^{-\mathrm{i}\sqrt{4\pi}\theta_{L,r}(x)} + \mathrm{e}^{\mathrm{i}\sqrt{4\pi}\theta_{R,r}(x)}\mathrm{e}^{\mathrm{i}\sqrt{4\pi}\theta_{L,r}(x)}] \tag{2}$$

$$S_r^-(x) = \frac{1}{\sqrt{2\pi a}}\left\{\frac{1}{2}[(-\mathrm{i})^x\mathrm{e}^{-\mathrm{i}\sqrt{\pi}\theta_r(x)} + (\mathrm{i})^x\mathrm{e}^{\mathrm{i}\sqrt{\pi}\theta_r(x)}]\right. \\ \left.\times\,[(\mathrm{i})^x\alpha_R\mathrm{e}^{\mathrm{i}\sqrt{\pi}(\theta_r(x)-\phi_r(x))} + (-\mathrm{i})^x\alpha_L\mathrm{e}^{-\mathrm{i}\sqrt{\pi}(\theta_r(x)+\phi_r(x))}]\right\} \tag{3}$$

and

$$S_r^+ = [S_r^-(x)]^\dagger. \tag{4}$$

The set of equations (2)–(4) are similar to those given in references [9, 21], the only difference lying in the global Majorana fermions which are chain independent. We substitute equations (2)–(4) into equation (1) and obtain the bosonized version of the three-chain problem. The bosonized Hamiltonian is further simplified when we replace $\theta_r(x)$ and $\phi_r(x)$ by the set of variables $\theta_\pm(x) = (1/\sqrt{2})(\theta_{-1}(x) \pm \theta_1(x))$, $\phi_\pm(x) = (1/\sqrt{2})(\phi_{-1}(x) \pm \phi_1(x))$, $\theta_0(x)$, and $\phi_0(x)$. This set of variables is in agreement with the arbitrary choice that the pairs are formed between the chains $r = 1$ and $r = -1$. Since the modes $\theta_\pm(x)$ and $\phi_\pm(x)$ will be integrated out, the effective Hamiltonian will depend on the variables $\theta_0(x)$ and $\phi_0(x)$ which will become the low-energy modes for the three-chain problem. The Hamiltonian equation (1) is expressed in terms of the fields $\theta_+(x)$, $\theta_-(x)$, $\theta_0(x)$ and their duals $\phi_+(x)$, $\phi_-(x)$, $\phi_0(x)$:

$$H = \int \mathrm{d}x\ h \qquad h = h_+ + h_- + h_0 + V. \tag{5}$$

The Hamiltonians h_+ and h_- are identical with those obtained in reference [12] in the limit $J_\perp/J_\parallel \longrightarrow 0$. The two chains $r = +1$ and $r = -1$ are described by the symmetric and antisymmetric Hamiltonians h_+ and h_-. h_0 represents the middle chain described in terms of θ_0 and ϕ_0. V represents the coupling between the modes '+','−', and '0'. We start with the two chains $r = 1$ and $r = -1$. Using the method described in reference [12] we obtain the symmetric and antisymmetric Hamiltonians h_+ and h_-. The Hamiltonians h_+ and h_- are given in terms of the dimensionless exchange parameters $\hat{J}_\perp$ and $\hat{J}_\parallel$. Following reference [12] we rescale the fields via $\theta_\pm \longrightarrow \sqrt{1/2}\theta_\pm$ and $\phi_\pm \longrightarrow \sqrt{2}\phi_\pm$. As a result, we obtain the Hamiltonians $h_\pm$:

$$h_+ = \frac{v^+}{2}[(\partial_x\phi_+)^2 + (\partial_x\theta_+)^2] - \frac{\hat{J}_\perp}{2(\pi a)^2}\cos(\sqrt{4\pi\epsilon_+}\theta_+) \tag{6}$$

$$h_- = \frac{v^-}{2}[(\partial_x\phi_-)^2 + (\partial_x\theta_-)^2] + \frac{\hat{J}_\perp}{2(\pi a)^2}\cos(\sqrt{4\pi\epsilon_-}\theta_-) \\ + \frac{\hat{J}_\perp}{(\pi a)^2}\cos\left(\sqrt{\frac{4\pi}{\epsilon_-}}\phi_-\right)[1 - \cos(\sqrt{4\pi\epsilon_-}\theta_-)]. \tag{7}$$

The set of equations (6) and (7) depends on the dimensionless velocities $v^{\pm}$ and the parameters $\epsilon_{\pm}$:

$$\epsilon_{\pm} = \sqrt{1 \pm \frac{1}{\pi^2}\left(\frac{\hat{J}_{\perp}}{\hat{J}_{\parallel}}\right)}. \tag{8}$$

$\epsilon_{\pm}$ and $v^{\pm}$ depend weakly on the ratio $\hat{J}_{\perp}/\hat{J}_{\parallel}$. In the limit $\hat{J}_{\perp}/\hat{J}_{\parallel} \longrightarrow 0$ we obtain $\epsilon_{\pm} \sim 1$ and $v^{\pm} \sim (\pi/2)\hat{J}_{\parallel}$. In this limit equations (6) and (7) are identical to those obtained in reference [12]. The bosonized form for the chain $r = 0$ is given by [25]

$$h_0 = \frac{v^0}{2}[(\partial_x\phi_0)^2 + (\partial_x\theta_0)^2] + \frac{\hat{J}_{\parallel}}{2(\pi a)^2}\cos(\sqrt{16\pi\epsilon_0}\theta_0) \tag{9}$$

where v^0 and ϵ_0 are given by [25]

$$v^0 \simeq \frac{\pi}{2}\hat{J}_{\parallel} \qquad \epsilon_0 = \sqrt{\frac{1-b}{1+b}} < 1 \qquad b = \frac{2/\pi}{1+1/\pi}. \tag{10}$$

Next, we represent the coupling between the modes '+', '−' and the chain $r = 0$ by $V \equiv V_1 + V_2$:

$$V_1 = \frac{\hat{J}_{\perp}}{\pi}\sqrt{\epsilon_0\epsilon_+}(\partial_x\theta_0)(\partial_x\theta_+) + \frac{2\hat{J}_{\perp}}{(\pi a)^2}\sin(\sqrt{4\pi\epsilon_0}\theta_0)[\sin(\sqrt{\pi\epsilon_+}\theta_+)\cos(\sqrt{\pi\epsilon_-}\theta_-)]$$
$$+ \frac{\hat{J}_{\perp}}{(\pi a)^2}\cos\left(\sqrt{\frac{\pi}{\epsilon_0}}\phi_0 - \sqrt{\frac{\pi}{\epsilon_+}}\phi_+\right)\cos\left(\sqrt{\frac{\pi}{\epsilon_-}}\phi_-\right) \tag{11}$$

$$V_2 = -\frac{\hat{J}_{\perp}}{(\pi a)^2}\left\{\cos\left(\sqrt{\frac{\pi}{\epsilon_0}}\phi_0 + \sqrt{4\pi\epsilon_0}\theta_0 - \sqrt{\frac{\pi}{\epsilon_+}}\phi_+ - \sqrt{\pi\epsilon_+}\theta_+\right)\cos\left(\sqrt{\frac{\pi}{\epsilon_-}}\phi_- + \sqrt{\pi\epsilon_-}\theta_-\right)\right.$$
$$+ \cos\left(\sqrt{\frac{\pi}{\epsilon_0}}\phi_0 - \sqrt{4\pi\epsilon_0}\theta_0 - \sqrt{\frac{\pi}{\epsilon_+}}\phi_+ + \sqrt{\pi\epsilon_+}\theta_+\right)$$
$$\left.\times \cos\left(\sqrt{\frac{\pi}{\epsilon_-}}\phi_- + \sqrt{\pi\epsilon_-}\theta_-\right)\right\}. \tag{12}$$

In obtaining equations (11) and (12) we have used the rescaled form of the fields $\theta_{\pm}$ and $\phi_{\pm}$. We see that V depends on $\epsilon_{\pm}$ and thus on the ratio $\hat{J}_{\perp}/\hat{J}_{\parallel}$.

3. The two-chain case

This problem has been considered in reference [12]. We take $V = 0$ and obtain that the two-chain problem is given by $h_+ + h_-$. Using the sine–Gordon scaling equations we can compute the gaps for the symmetric and antisymmetric modes. In reference [12] the gaps have been evaluated for $\epsilon_+ = \epsilon_- = 1$. The justification for making this approximation is provided by the fact that an exact solution can be obtained by mapping the problem to Majorana fermions. The model h_+ in equation (6) is in the disordered phase for $\hat{J}_{\perp} > 0$ with a positive mass $M_+ > 0$. We reduce the cut-off Λ to Λe^{-l} and find the dimensionless gap $\hat{M}_+$ for the Hamiltonian h_+:

$$\hat{M}_+(\lambda_+) = e^{-l}\hat{M}_+(\lambda_+(l)).$$

The gap depends on the coupling constant

$$\lambda_+ \stackrel{\text{def}}{=} \frac{2}{\pi}\frac{\hat{J}_{\perp}}{\hat{J}_{\parallel}}.$$

Using the sine–Gordon scaling we find that the coupling constant λ_+ obeys

$$\lambda_+(l) = \lambda_+ e^{(2-\epsilon_+)l}.$$

The parameter ϵ_+ is given in equation (8). Taking $\lambda_+(l) \simeq 1$ and substituting l into the gap scaling equation, we find that

$$\hat{M}_+ = \left(\frac{2\hat{J}_\perp}{\pi \hat{J}_\parallel}\right)^{1/(2-\epsilon_+)} \sim \left(\frac{2\hat{J}_\perp}{\pi \hat{J}_\parallel}\right)\left[1 - \frac{\hat{J}_\perp}{\pi^2 \hat{J}_\parallel} \log\left(\frac{\hat{J}_\perp}{\hat{J}_\parallel}\right)\right]. \tag{13}$$

When $\epsilon_+ = 1$ we find the same result as in reference [12].

The study of the h_--mode in equation (7) is more complicated, since for $\hat{J}_\perp > 0$ the coefficients of $\cos(\sqrt{4\pi\epsilon_-}\theta_-)$ and $\cos(\sqrt{4\pi/\epsilon_-}\phi_-)$ are positive; this describes a situation below the phase transition. Here we compute the mass gaps M_- and $\tilde{M}_-$ for the field θ_- and its dual field ϕ_- respectively. When $\epsilon_- = 1$, the mapping to an Ising model is possible, showing that one field is in the disordered phase, and the other one in the ordered phase. Performing a similar analysis to the one performed for h_+ gives us

$$|\hat{M}_-| = \left(\frac{2\hat{J}_\perp}{\pi \hat{J}_\parallel}\right)^{1/(2-\epsilon_+)} \sim \left(\frac{2\hat{J}_\perp}{\pi \hat{J}_\parallel}\right)\left[1 - \frac{\hat{J}_\perp}{\pi^2 \hat{J}_\parallel} \log\left(\frac{\hat{J}_\perp}{\hat{J}_\parallel}\right)\right] \tag{14}$$

and for the mass for the dual one $\tilde{M}_-$

$$|\hat{\tilde{M}}_-| = \left(\frac{2\hat{J}_\perp}{\pi \hat{J}_\parallel}\right)^{1/(2-1/\epsilon_-)} \sim \left(\frac{2\hat{J}_\perp}{\pi \hat{J}_\parallel}\right)\left[1 + \frac{1}{\pi^2} \log\left(\frac{\hat{J}_\perp}{\hat{J}_\parallel}\right)\right] \tag{15}$$

In obtaining the results in equations (14) and (15) we have neglected the term

$$\cos\left(\sqrt{\frac{4\pi}{\epsilon_-}}\phi_-\right)\cos(\sqrt{4\pi\epsilon_-}\theta_-).$$

This term has a marginal dimension $2 - \epsilon_- - 1/\epsilon_- \simeq 0$ corresponding to the sine–Gordon scaling equation, and can be neglected relative to the relevant terms $\cos(\sqrt{4\pi\epsilon_-}\theta_-)$, which has a positive dimension $2 - \epsilon_- > 0$, and $\cos(\sqrt{4\pi/\epsilon_-}\phi_-)$, with a dimension $2 - 1/\epsilon_- > 0$. For the remainder of this paper we will neglect the marginal term $\cos(\sqrt{4\pi/\epsilon_-}\phi_-)\cos(\sqrt{4\pi\epsilon_-}\theta_-)$. We will see that a dimensionality argument will justify the neglect of V_2 (equation (12)). As a result we need only to study the effect of V_1 given by equation (11).

4. The effective action for the massless mode—the three-chain case

For the three-chain case the presence of the massive excitations of h_+ and h_- (equations (6) and (7)) facilitates the derivation of an effective action

$$\tilde{h}_0, \tilde{h}_0 \stackrel{\text{def}}{=} h_0 + V_{eff}.$$

h_0 is given by equation (9) and V_{eff} is the effective action obtained by projecting out the massive modes. $\tilde{h}_0$ is obtained in the following way. We introduce an evolution operator $U(t, 0)$:

$$U(t,0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n \ \hat{T}[V(t_1)\cdots V(t_n)]. \tag{16}$$

$V(t)$ is the coupling term given by equations (11) and 12 in the interaction picture. The effective action is obtained by taking the trace over the modes $\{+, -\}$:

$$\langle U(t,0)\rangle \stackrel{\text{def}}{=} \mathrm{Tr}_{\{+,-\}}\, U(t,0) \tag{17}$$

$$\mathrm{Tr}_{\{0,+,-\}}[\mathrm{e}^{-\mathrm{i}Ht}] = \mathrm{Tr}_{\{0\}}[\mathrm{e}^{-\mathrm{i}H_0 t}\langle U(t,0)\rangle] \tag{18}$$

where

$$H_0 = \int \mathrm{d}x\ h_0.$$

Equations (17) and (18) define the effective action $\tilde{h}_0$:

$$\tilde{h}_0 = h_0 + V_{eff} \qquad V_{eff} = \sum_{n=1} \frac{1}{n!} V^{(n)} \tag{19}$$

where the $V^{(n)}$ are the cumulant terms obtained from V:

$$V^{(1)} = \langle V\rangle_{\{+,-\}} \qquad V^{(2)} = \langle V^2\rangle_{\{+,-\}} - \langle V\rangle^2_{\{+,-\}} \tag{20}$$

and so on. $V^{(1)}$ is the first cumulant and $V^{(2)}$ the second one. The crucial fact in the computation of the cumulants in equation (19) is that the modes θ_+, θ_-, and ϕ_- are massive. As a result,

$$\langle \sin(\sqrt{\pi\epsilon_+}\theta_+)\rangle = \left\langle \sin\left(\sqrt{\frac{\pi}{\epsilon_+}}\phi_+\right)\right\rangle = \left\langle \cos\left(\sqrt{\frac{\pi}{\epsilon_-}}\phi_-\right)\right\rangle = 0 \tag{21}$$

from which we conclude that the first moment vanishes:

$$V^{(1)} = \langle V\rangle_{\{+,-\}} = 0. \tag{22}$$

The nth-order cumulant scales like $(J_\perp)^n$. Such terms will induce sine–Gordon terms of the form $\cos(n\sqrt{4\pi\epsilon_0}\theta_0)$ or $\cos(n\sqrt{4\pi/\epsilon_0}\phi_0)$. They have negative scaling dimensions for $n \geqslant 3$. So these terms are irrelevant and will be ignored. The only effect of the $n \geqslant 3$ terms is to renormalize the coupling constants of the relevant terms. Keeping only terms of the order $n = 2$, from $V^{(2)} = V_1^{(2)}$ (the $V_2^{(2)}$-terms are irrelevant according to the sine–Gordon naive scaling dimension) we obtain

$$\tilde{H}_0 = \int \mathrm{d}x\ \tilde{h}_0(x) \tag{23}$$

$$\begin{aligned}\tilde{h}_0(x) = h_0(x) &+ \frac{\hat{J}_\perp^2}{\mathrm{i}} \int \mathrm{d}x' \int_0^t \mathrm{d}t'\ \frac{\epsilon_0\epsilon_1}{\pi^2} F(x',t')\, \partial_x\theta_0(x,t)\, \partial_x\theta_0(x+x',t-t') \\ &+ \frac{1}{2\mathrm{i}}\left(\frac{2J_\perp}{\pi a}\right)^2 \int \mathrm{d}x' \int_0^t \mathrm{d}t'\ \Bigg\{ R(x',t') \\ &\times \frac{1}{2}[\cos(\sqrt{4\pi\epsilon_0}(\theta_0(x,t) - \theta_0(x+x',t-t'))) \\ &- \cos(\sqrt{4\pi\epsilon_0}(\theta_0(x,t) + \theta_0(x+x',t-t')))] \\ &+ G(x',t')\cos\left[\sqrt{\frac{\pi}{\epsilon_0}}(\phi_0(x,t) - \phi_0(x+x',t-t'))\right]\Bigg\}\end{aligned} \tag{24}$$

where the functions $F(x',t')$, $R(x',t')$, and $G(x',t')$ are obtained from the correlation functions of $\theta_+, \theta_-, \phi_+$, and ϕ_-:

$$F(x',t') = \langle \partial_x\theta_+(x,t)\, \partial_x\theta_+(x+x',t-t')\rangle_{\{+,-\}}. \tag{25}$$

Due to the fact that the field θ_+ has a mass $M_+ > 0$ (see equation (13)), the function F will create higher-order derivatives which are irrelevant and will be ignored. The only relevant correlations are determined by the functions $R(x', t')$ and $G(x', t')$:

$$R(x', t') = \langle :\sin[\sqrt{\pi\epsilon_+}\theta_+(x', t')]::\sin[\sqrt{\pi\epsilon_+}\theta_+(0)]:\rangle_{\{+\}} \times \langle :\cos[\sqrt{\pi\epsilon_-}\theta_-(x', t')]::\cos[\sqrt{\pi\epsilon_-}\theta_-(0)]:\rangle_{\{-\}} \tag{26}$$

$$G(x', t') = \left\langle :\cos\left[\sqrt{\frac{\pi}{\epsilon_+}}\phi_+(x', t')\right]::\cos\left[\sqrt{\frac{\pi}{\epsilon_+}}\phi_+(0)\right]:\right\rangle_{\{+\}} \times \left\langle :\cos\left[\sqrt{\frac{\pi}{\epsilon_-}}\phi_-(x', t')\right]::\cos\left[\sqrt{\frac{\pi}{\epsilon_+}}\phi_-(0)\right]:\right\rangle_{\{-\}} \tag{27}$$

where :: represents the normal order and $\langle\ \rangle_{\{\pm\}}$ represents the expectation value with respect to the Hamiltonians $h_\pm$ given in equations (6) and (7). Since the fields $\theta_+, \theta_-, \phi_+$, and ϕ_- are massive, it follows that the functions R and G decay exponentially at large distances. It is this fact which allows us to perform a derivative expansion resulting in a sine–Gordon model which is, in the massless phase, similar to the situation for the single Heisenberg chain. An explicit calculation is easily performed using the mapping of h_+ and h_- to Majorana fermions and the two-dimensional Ising model.

5. Computation of the effective action for the three-chain problem using the real Majorana fermion representation

The purpose of this section is to show that for $\epsilon_+ \sim \epsilon_- \sim 1$ one can find an exact form of the functions $R(x', t')$ and $G(x', t')$ in terms of the known correlation functions of the two-dimensional Ising model [15, 23]. We perform this calculation in the following steps.

(a) Express the Majorana fermions as a function of the bosonic fields θ_r and ϕ_r.
(b) Identify h_+ and h_- each with two Majorana models.
(c) Identify each Majorana model with an Ising model (in the ordered or disordered phase).
(d) Compute R and G using the Ising language.

We introduce two pairs of real Majorana spinors:

$$\hat{\chi}_r(x) = \begin{pmatrix} \chi_{R,r}(x) \\ \chi_{L,r}(x) \end{pmatrix} \qquad \hat{\eta}_r(x) = \begin{pmatrix} \eta_{R,r}(x) \\ \eta_{L,r}(x) \end{pmatrix} \qquad r = +, -$$

Following reference [16] we have

$$\chi_{R,r}(x) = \frac{1}{\sqrt{\pi a}}:\cos[\sqrt{\pi}(\theta_r(x) - \phi_r(x))]: \qquad \chi_{L,r}(x) = \frac{1}{\sqrt{\pi a}}:\cos[\sqrt{\pi}(\theta_r(x) + \phi_r(x))]: \tag{28}$$

and

$$\eta_{R,r}(x) = \frac{1}{\sqrt{\pi a}}:\sin[\sqrt{\pi}(\theta_r(x) - \phi_r(x))]: \qquad \eta_{L,r}(x) = \frac{1}{\sqrt{\pi a}}:\sin[\sqrt{\pi}(\theta_r(x) + \phi_r(x))]:. \tag{29}$$

Using equations (28) and (29) we map h_+ and h_- into Majorana Hamiltonians [12]. Each Majorana Hamiltonian is equivalent to a 1D quantum Ising model [24]. For each spinor we introduce an order variable and a disorder variable. For the spinor $\hat{\chi}_r(x)$ we introduce $\sigma_r(x)$, the order variable, and $\mu_r(x)$, the disorder one. For $\hat{\eta}_r(x)$ we introduce $\tau_r(x)$, the order field, and $\rho_r(x)$, the disorder field. Following reference [12] we find for $\epsilon_\pm \simeq 1$ the results for h_r, $r = +, -$:

$$h_r = h_r(\hat{\chi}) + h_r(\hat{\eta}) \tag{30}$$

$$h_r(\hat{\chi}) = \frac{v^{(r)}}{2}(\chi_{R,r}(x)(-\mathrm{i}\,\partial_x)\chi_{R,r}(x) + \chi_{L,r}(x)(\mathrm{i}\,\partial_x)\chi_{L,r}(x)) - \mathrm{i}m_r\chi_{R,r}(x)\chi_{L,r}(x) \tag{31}$$

$$h_r(\hat{\eta}) = \frac{v^{(r)}}{2}(\eta_{R,r}(x)(-\mathrm{i}\,\partial_x)\eta_{R,r}(x) + \eta_{L,r}(x)(\mathrm{i}\,\partial_x)\eta_{L,r}(x)) - \mathrm{i}\tilde{m}_r\eta_{R,r}(x)\eta_{L,r}(x). \tag{32}$$

Using the Ising language we relate the external field λ_r to the mass m_r and find

$$h_r(\hat{\chi}) \equiv h_r(\sigma_r) = -\lambda_r \sum_x \sigma_r^z(x) - \sum_x \sigma_r(x)\sigma_r(x+1) \tag{33}$$

with $m_r = (\lambda_r - 1)\Lambda$ and

$$h_r(\hat{\eta}) \equiv h_r(\tau_r) = -\tilde{\lambda}_r \sum_x \tau_r^z(x) - \sum_x \tau_r(x)\tau_r(x+1) \tag{34}$$

with $\tilde{m}_r = (\tilde{\lambda}_r - 1)\Lambda$. When $m_r > 0$ ($\tilde{m}_r > 0$) the Ising model is in the disordered phase and obeys $\langle\sigma_r\rangle = 0$, $\langle\mu_r\rangle \neq 0$ ($\langle\tau_r\rangle = 0$, $\langle\rho_r\rangle \neq 0$). For $m_r < 0$ ($\tilde{m}_r < 0$) the Ising model is in the ordered phase $\langle\sigma_r\rangle \neq 0$, $\langle\mu_r\rangle = 0$ ($\langle\tau_r\rangle \neq 0$, $\langle\rho_r\rangle = 0$). The computation of V_{eff} in equation (19) is done using the Ising variables:

$$:\sin[\sqrt{\pi}\theta_r(x)]: = \sigma_r(x)\tau_r(x) \tag{35}$$

$$:\cos[\sqrt{\pi}\theta_r(x)]: = \mu_r(x)\rho_r(x) \tag{36}$$

$$:\sin[\sqrt{\pi}\phi_r(x)]: = \mu_r(x)\tau_r(x) \tag{37}$$

$$:\cos[\sqrt{\pi}\phi_r(x)]: = \sigma_r(x)\rho_r(x). \tag{38}$$

We remark that the identification given by equations (35)–(38) holds only for $\epsilon_\pm = 1$. Replacing $\epsilon_\pm \sim 1$ by $\epsilon_\pm = 1$ means that we neglect corrections of the order $(\hat{J}_\perp/\hat{J}_\parallel)^2$ in V_{eff} which scales like $\hat{J}_\perp^2$ (see equation (24)). From equations (6) and (7) we identify, in agreement with reference [12],

$$m_+ = \tilde{m}_+ = m_- = \frac{\hat{J}_\perp}{\pi a} \qquad \tilde{m}_- = -3m_+. \tag{39}$$

The sign of the Majorana masses depends on the sign of $\hat{J}_\perp$. For $\hat{J}_\perp > 0$, $m_+ = \tilde{m}_+ > 0$ and the Ising models are in the disordered phase, $\langle\sigma_+\rangle = \langle\tau_+\rangle = 0$ and $\langle\mu_+\rangle = \langle\rho_+\rangle \neq 0$. The Ising mode $r = -$ has $m_- > 0$ and $\tilde{m}_- < 0$, resulting in $\langle\sigma_-\rangle = \langle\rho_-\rangle = 0$ and $\langle\mu_-\rangle = \langle\tau_-\rangle \neq 0$. When $\hat{J}_\perp < 0$ the disordered phase is mapped into the ordered one, $\sigma \longrightarrow \mu$, $\tau \longrightarrow \rho$ and $\mu \longrightarrow \sigma$, $\rho \longrightarrow \tau$. The crucial point is that the first cumulant vanishes, $V^{(1)} = 0$, irrespective of the sign of $\hat{J}_\perp$. Qualitatively the result for the higher-order cumulants is independent of the sign of $\hat{J}_\perp$. Quantitatively the values of $R(x', t)$ and $G(x', t)$ are different. For $\hat{J}_\perp > 0$ we have three Ising models in the disordered phase and one in the ordered phase. When $\hat{J}_\perp < 0$ three Ising models are in the ordered phase and one is in the disordered phase.

We compute $R(x', t)$ and $G(x', t)$ for $\epsilon_\pm = 1$ and find $R(x', t) = G(x', t)$. Using the Ising variables we find

$$R(x', t') = \langle\sigma_+(x', t')\sigma_+(0)\rangle_{m_+}\langle\tau_+(x', t')\tau_+(0)\rangle_{\tilde{m}_+} \times \langle\mu_-(x', t')\mu_-(0)\rangle_{m_-}\langle\rho_-(x', t')\rho_-(0)\rangle_{\tilde{m}_-}. \tag{40}$$

For $\hat{J}_\perp > 0$, $R(x', t') \stackrel{\text{def}}{=} R^{(+)}(x', t')$, equation (40) is simplified. $\langle \mu_- \rangle \neq 0$, resulting in $\langle \mu_-(x', t')\mu_-(0)\rangle \simeq [\langle \mu_- \rangle_{m_-}]^2$. Since $m_+ = \tilde{m}_+ > 0$, the σ_+-correlation is equal to the τ_+-correction. Since $\tilde{m}_- < 0$, we have

$$\langle \rho_-(x', t')\rho_-(0)\rangle_{\tilde{m}_-} = \langle \tau_-(x', t')\tau_-(0)\rangle_{-\tilde{m}_-}.$$

As a result, equation (40) takes the following form for $\hat{J}_\perp > 0$:

$$R^{(+)}(x', t') = [\langle \sigma(x', t')\sigma(0)\rangle_m]^2[\langle \sigma \rangle_{-m}]^2 \langle \sigma(x', t')\sigma(0)\rangle_{3m}. \tag{41}$$

In equation (41) we have used an Ising model with $m = \hat{J}_\perp/\pi a > 0$. When $\hat{J}_\perp < 0$, we define $R(x', t') \stackrel{\text{def}}{=} R^{(-)}(x', t')$ and obtain from equation (40) instead of equation (41)

$$R^{(-)}(x', t') = [\langle \sigma \rangle_m]^4 \langle \sigma(x', t')\sigma(0)\rangle_{-m}[\langle \sigma \rangle_{3m}]^2 \tag{42}$$

where $m = \hat{J}_\perp/\pi a < 0$. The explicit results in equations (41) and (42) depend on the two-dimensional Ising correlation function [23] $C(s)$:

$$C(s = \sqrt{x^2 - v^2 t^2}) \stackrel{\text{def}}{=} \langle \sigma(x, t)\sigma(0)\rangle. \tag{43}$$

This correlation function is computed in the disordered phase and depends on the mass

$$M \equiv \frac{m}{v^*} = \left(\frac{|\hat{J}_\perp|}{\hat{J}_\parallel}\right)\left(\frac{1}{\pi a}\right).$$

We obtain

$$C(s) = \begin{cases} \sqrt{\dfrac{2}{\pi M|s|}}\mathrm{e}^{-Ms} & M|s| \longrightarrow \infty \\ \dfrac{1}{(M|s|)^{1/4}}\left[1 - \dfrac{1}{2}M|s| \log\left(\dfrac{1}{2}M|s|\right)\right] & M|s| \longrightarrow 0. \end{cases} \tag{44}$$

In the ordered phase, we take $-M$ and obtain [14]

$$\langle \sigma \rangle_{-M} = (Ma)^\beta \qquad \frac{1}{8} \leqslant \beta \leqslant \frac{1}{2} \qquad Ma = \frac{1}{\pi}\frac{|\hat{J}_\perp|}{\hat{J}_\parallel} \tag{45}$$

where $\beta = \beta_c = 1/8$ is the critical exponent of the Ising model, and $\beta = 1/2$ is the mean-field exponent.

The critical region corresponds to $\hat{J}_\perp/\hat{J}_\parallel \longrightarrow 0$. This region is characterized by $\beta = \beta_c = 1/8$ and the Ising correlation function $C(s) \sim 1/|s|^{1/4}$. The functions $R^{(+)}(x', t')$ and $R^{(-)}(x', t')$ take the following forms in the critical region, $Ma = (1/\pi)|\hat{J}_\perp|/\hat{J}_\parallel \longrightarrow 0$:

$$R^{(+)}(s) \sim \frac{1}{3^{1/4}}\frac{(Ma)^{1/4}}{(M|s|)^{3/4}} \qquad R^{(-)}(s) \sim 3^{1/4}\frac{(Ma)^{1/2}}{(M|s|)^{1/4}}. \tag{46}$$

The functions $R^{(+)}(x', t')$ and $R^{(-)}(x', t')$ must be substituted in the effective Hamiltonian given in equation (24). Due to the long-range behaviour, no simple correspondence with the single spin-$\frac{1}{2}$ chain can be found. From equation (46) we see that the critical behaviours for $\hat{J}_\perp > 0$ and $\hat{J}_\perp < 0$ are different. The critical behaviour of the model given in equation (24) will be considered elsewhere. Next we concentrate our investigation on regions away from the critical region using equation (44) for $M|s| \longrightarrow \infty$, and $\beta \simeq 1/2$ in equation (45). We find from equations (44), (45), and (41)

$$R^{(+)}(s) \simeq \sqrt{\frac{1}{3}\left(\frac{2}{\pi}\right)^3}\frac{(Ma)^{2\beta}}{(M|s|)^{3/2}}\mathrm{e}^{-5sM}. \tag{47}$$

Correspondingly, from equation (42) we find for $\hat{J}_\perp < 0$

$$R^{(-)}(s) \simeq \sqrt{2}\pi \frac{(Ma)^{2\beta}(3Ma)^{2\beta}}{(M|s|)^{1/2}} \mathrm{e}^{-sM}. \tag{48}$$

We substitute equations (47) and (48) (for $\hat{J}_\perp < 0$) into the effective Hamiltonian $\tilde{h}_0$ in equation (24). We perform a Euclidean rotation for $R(x', t') \to R(x', \mathrm{i}\tau'), t' \to \mathrm{i}\tau'$ and perform a derivative expansion in equation (24). We make the substitutions $\partial_t\theta_0 = v^{(0)}\,\partial_x\phi_0$, $\partial_t\phi_0 = -v^{(0)}\,\partial_x\theta_0$ and keep only relevant terms in the expansion (we neglect higher-order derivatives). The effective Hamiltonian in equation (24) is replaced by

$$\tilde{h}_0 = \frac{v^0}{2}\left[((\partial_x\phi_0)^2 + (\partial_x\theta_0)^2) + c_1\left(\frac{2|\hat{J}_\perp|}{\pi\hat{J}_\parallel}\right)^{2(\gamma-1)}((\partial_x\theta_0)^2 - (\partial_x\phi_0)^2)\right]$$
$$+ \frac{\hat{J}_\parallel}{2(\pi a)^2}\left[1 - c_0\left(\frac{2|\hat{J}_\perp|}{\pi\hat{J}_\parallel}\right)^{2\gamma}\right]\cos(\sqrt{16\pi\epsilon_0}\theta_0) \tag{49}$$

where c_0, c_1, and ϵ_0 are numerical constants:

$$\epsilon_0 = \sqrt{\frac{1-1/\pi}{1+3/\pi}} \simeq 0.6 \qquad c_0 \simeq 1 \qquad c_1 \simeq 10^{-3}\left(\frac{1}{2\epsilon_0} - \epsilon_0\right) \simeq 2.4\times 10^{-4}.$$

The exponent γ replaces β:

$$\gamma = \begin{cases} \beta & \hat{J}_\perp > 0 \\ 2\beta & \hat{J}_\perp < 0. \end{cases} \tag{50}$$

From equations (49)–(50) we see that for $\hat{J}_\perp < 0$, $\gamma - 1 = 2\beta - 1 \simeq 0$, allowing us to neglect the term

$$c_1(2|\hat{J}_\perp|/\pi\hat{J}_\parallel)^{2(\gamma-1)}((\partial_x\theta_0)^2 - (\partial_x\phi_0)^2).$$

As a result, equation (49) takes the same form as h_0 in equation (9), where $\hat{J}_\parallel$ is replaced by $\hat{J}_\parallel(1 - c_0(2|\hat{J}_\perp|/\pi\hat{J}_\parallel)^2)$.

Next we consider the case of positive interchain coupling $\hat{J}_\perp > 0$ outside the critical region. We have for this case $\gamma - 1 = \beta - 1 \sim -1/2$ and $\gamma = \beta \simeq 1/2$. For this case the corrections in equation (49) cannot be ignored! We rescale the Hamiltonian in equation (49) via $\phi_0 \to \phi_0/\sqrt{K}$, $\theta_0 \to \theta_0\sqrt{K}$ where K is given by

$$K = \sqrt{\left[1 - c_1\left(\frac{2|\hat{J}_\perp|}{\pi\hat{J}_\parallel}\right)^{2(\gamma-1)}\right]\Big/\left[1 + c_1\left(\frac{2|\hat{J}_\perp|}{\pi\hat{J}_\parallel}\right)^{2(\gamma-1)}\right]} \qquad c_1 \simeq 2.4\times 10^{-4}. \tag{51}$$

The effective Hamiltonian in equation (49) is replaced for both cases by

$$\bar{h}_0 = \frac{v^0_{eff}}{2}[(\partial_x\phi_0)^2 + (\partial_x\theta_0)^2] + \frac{\hat{J}_{eff}}{2(\pi a)^2}\cos(\sqrt{16\pi K_{eff}}\theta_0) \tag{52}$$

where

$$K_{eff} \equiv \epsilon_0 K \qquad v^0_{eff} = v^0\sqrt{1 - \left(c_1\left(\frac{2|\hat{J}_\perp|}{\pi\hat{J}_\parallel}\right)^{2(\gamma-1)}\right)^2} \tag{53}$$

and $\hat{J}_{eff}$ is given by

$$\hat{J}_{eff} = \hat{J}_\parallel\left[1 - c_0\left(\frac{2|\hat{J}_\perp|}{\pi\hat{J}_\parallel}\right)^{2\gamma}\right]. \tag{54}$$

The effective Hamiltonian $\bar{h}_0$ in equation (52) describes massless excitations if the parameter K_{eff} (see equations (51) and (53)) obeys $1/2 \leqslant K_{eff} \leqslant 1$. Using equation (51) we find that this can be satisfied if $(2\hat{J}_\perp)/(\pi \hat{J}_\parallel) \geqslant 10^{-3}$. The limit $\hat{J}_\perp/\hat{J}_\parallel \to 0$ requires a renormalization group analysis. The difficulty lies in the fact that the derivative expansion is not valid (see the discussion following equation (46)).

As a result, the effective Hamiltonian in equation (52) describes massless excitations similar to those obtained for the spin-$\frac{1}{2}$ Heisenberg chain. It seems that only for $\hat{J}_\perp \longrightarrow 0$ are the behaviours for the positive and the negative signs different. For moderate values of $\hat{J}_\perp$ the qualitative behaviour is independent of the sign of $\hat{J}_\perp$.

6. Conclusion

Three coupled spin-$s = \frac{1}{2}$ AF chains have been investigated within the bosonization method. Projecting out the $s = 1$ massive modes of the two-chain problem, we find that the low-energy excitations of the three-chain problem are massless and are similar to those of the single $s = \frac{1}{2}$ AF chain conjectured in reference [1]. We find that the qualitative behaviour of the massless excitations in the three-spin-chain problem is independent of the sign of the interchain coupling $\hat{J}_\perp$. The results obtained are in agreement with those given in references [7] and [8] based on the coherent-state and semiclassical approximations for spins. The method introduced here can be used to study the doped striped cuprates in high-T_c materials, and can be extended to the study of many coupled chains.

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